Intro. to ODEs

Quiz 9 **Solutions**

1) Solve the general solution to the following differential equation.

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 4e^{-t}\ln(t) \qquad (t > 0)$$

The characteristic equation is $r^2+2r+1 = (r+1)^2 = 0$ with roots $r = -1, -1$. So, the homogeneous solution is

$$x_H(t) = c_1e^{-t} + c_2te^{-t}.$$

To find a particular solution, we implement variation of parameters.

$$\begin{aligned}
x_P(t) &= v_1(t)e^{-t} + v_2(t)te^{-t} \\
W[x_1, x_2](t) &= \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t} \\
v_1(t) &= -\int \frac{4te^{-2t}\ln(t)}{e^{-2t}}\ dt = -4\int t\ln(t)\ dt = t^2 - 2t^2\ln(t) \\
v_2(t) &= \int \frac{4e^{-2t}\ln(t)}{e^{-2t}}\ dt = 4\int \ln(t)\ dt = 4t\ln(t) - 4t \\
x_P(t) &= \left(t^2 - 2t^2\ln(t)\right)e^{-t} + (4t\ln(t) - 4t)\,te^{-t} \\
&= -3t^2e^{-t} + 2t^2e^{-t}\ln(t)
\end{aligned}$$

So, the general solution is

$$x(t) = c_1e^{-t} + c_2te^{-t} - 3t^2e^{-t} + 2t^2e^{-t}\ln(t).$$

2) Solve the general solution to the following differential equation.

$$\frac{d^2x}{dt^2} + x = \tan(t)\sec(t)$$

The characteristic equation is $r^2 + 1 = 0$ with roots $r = \pm i$. So, the homogeneous solution is

$$x_H(t) = c_1\cos(t) + c_2\sin(t).$$

To find a particular solution, we implement variation of parameters.

$$\begin{aligned}
x_P(t) &= v_1(t)\cos(t) + v_2(t)\sin(t) \\
W[x_1, x_2](t) &= \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = 1 \\
v_1(t) &= -\int \frac{\tan(t)\sec(t)\sin(t)}{1}\ dt = -\int \tan^2(t)\ dt = -\int \left(\sec^2(t) - 1\right)\ dt = -\tan(t) + t \\
v_2(t) &= \int \frac{\tan(t)\sec(t)\cos(t)}{1}\ dt = \int \tan(t)\ dt = -\ln|\cos(t)| \\
x_P(t) &= (t - \tan(t))\cos(t) - \ln|\cos(t)|\sin(t) \\
&= t\cos(t) - \sin(t) - \ln|\cos(t)|\sin(t)
\end{aligned}$$

So, the general solution is

$$x(t) = c_1 \cos(t) + c_2 \sin(t) + t\cos(t) - \ln|\cos(t)|\sin(t)$$

where we drop the extra $-\sin(t)$ term from the particular solution since it can absorbed into the homogeneous part of the solution.

3) A mass of size $m = 1$ is attached to a spring with strength $k = 10$. The coefficient of resistance for the system is $b = 2$, and an external force $F(t) = 10\cos(4t)$ is acting on the system. Write down the differential equation that models the position of the mass $x = x(t)$ as measured from equilibrium. Find the *steady–state* portion of the solution (i.e. the particular solution).

The differential equation for this system is

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 10\cos(4t).$$

Since this is a damped oscillator and we are only asked for the steady–state solution $x_P(t)$, we don't need to find the homogeneous solution in this case (as it definitely won't be anything like the periodic forcing term). We can use Undetermined Coefficients for the particular solution.

$$\begin{aligned} x_P(t) &= A\cos(4t) + B\sin(4t) \\ x_P'(t) &= 4B\cos(4t) - 4A\sin(4t) \\ x_P''(t) &= -16A\cos(4t) - 16B\sin(4t) \end{aligned}$$

Substituting these into the differential equation gives

$$\big(-6A + 8B\big)\cos(4t) + \big(-6B - 8A\big)\sin(4t) = 10\cos(4t).$$

So, we need

$$\begin{aligned} -6A + 8B &= 10 \\ -6B - 8A &= 0. \end{aligned}$$

This gives $A = -\dfrac{3}{5}$ and $B = \dfrac{4}{5}$, and the steady–state solution is

$$x_P(t) = -\frac{3}{5}\cos(4t) + \frac{4}{5}\sin(4t).$$